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Grassmann Green's functions

Silvio J Rabello†, Luiz C de Albuquerque‡ and Arvind N Vaidya

Instituto de Física, Universidade Federal do Rio de Janeiro, Rio de Janeiro RJ, Caixa Postal 68.528-CEP 21945-970, Brazil

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Abstract. The Green function for an oscillator with anticommuting degrees of freedom is obtained by extending two operator methods to the case of a Grassmann algebra: the dynamical-group approach and the Dirac–Schwinger method. In both methods, we verify that the canonical anticommutation relations are responsible for the fact that this system has a negative-dimensional behaviour.

1. Introduction

Grassmann variables are nowadays of widespread use in theoretical physics, for example in the description of fermions by the functional methods of quantum field theory, in supersymmetry, in the Bechi–Rouet–Stora–Tyutin (BRST) theory of classical- and quantum-constrained systems, to mention just a few [1]. In order to explore the role of anticommuting coordinates in the supersymmetric version of the Poincaré group, Finkelstein and Villasante [2] introduced a Grassmann generalization of ordinary quantum mechanics, where the degrees of freedom are considered to be elements of a Grassmann algebra. As an application, they studied an N-dimensional Grassmann oscillator (GO), that, in contrast to the usual first-order Grassmann models of supersymmetry, is of second order in time derivatives. This system is related to the usual harmonic oscillator (HO) by the change $N \to -N$ as pointed out by Dunne and Halliday [3] using the path-integral representation for the Green function and the properties of the Gaussian Berezin integral. This fact allows one to understand the Grassmann coordinates as negative-dimensional degrees of freedom and use them to explore the realm of the negative-dimensional groups [4]. In [5], it was found that at finite temperature, although described by Grassmann numbers, the negative-dimensional oscillator displays a bosonic behaviour, due to a hidden BRST symmetry in the classical mechanics of the usual HO.

In this paper, we follow [2] and obtain the Green function for the GO by extending two operator methods of ordinary quantum mechanics to anticommuting variables: the dynamical-group method and the Dirac–Schwinger method. In the following sections, we introduce the notation and explain both methods by explicit calculation of the Green function.

† E-mail: rabello@ntsu1.if.ufrj.br

‡ E-mail: farina@vms1.nce.ufrj.br

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2. The Grassmann oscillator

The Grassmann version of ordinary quantum mechanics, studied by Finkelstein and Villasante [2], is described by the coordinate $\hat{q}_a$ and conjugate momentum $\hat{p}_a$ operators ($a = 1, \ldots, N$) that obey the canonical anticommutation relations ($\hbar = 1$)

$$[\hat{q}_a, \hat{p}_b]_+ = -\mathrm{i}\delta_{ab} \qquad [\hat{q}_a, \hat{q}_b]_+ = 0 \qquad [\hat{p}_a, \hat{p}_b]_+ = 0. \tag{1}$$

They considered an oscillator with the Hamiltonian operator given by (summation convention assumed)

$$\hat{H} = \tfrac{1}{2}(-\hat{p}_a C^{-1}_{ab} \hat{p}_b + \omega^2 q_a C_{ab} q_b) \tag{2}$$

where C is a Hermitian antisymmetric matrix. As an antisymmetric matrix has an inverse only for even dimensions, we are restricted to even values of N.

To describe the Green function for the Schrödinger equation with the above $\hat{H}$, we introduce a basis of $\hat{q}_a$ eigenvectors $|q\rangle$ where the eigenvalues q_a are elements of a Grassmann algebra and the normalization is of the Berezin delta-function type [1]: $\langle q'|q\rangle = \delta(q' - q)$. With the above definitions, we have for the Green function

$$\langle q', t|q, 0\rangle = \langle q'|\mathrm{e}^{-\mathrm{i}\hat{H}t}|q\rangle. \tag{3}$$

Among the several ways of obtaining the above matrix element, in the following sections, we focus on two operator methods: the first in the Schrödinger picture and the second in the Heisenberg picture.

3. The dynamical-group method

In the Schrödinger picture, we can use the coordinate representation $\hat{q}_a = q_a$ and $\hat{p}_a = -\mathrm{i}\frac{\partial}{\partial q_a}$ (hereafter, all derivatives act by the left), so that (3) now reads

$$\langle q'|\mathrm{e}^{-\mathrm{i}\hat{H}t}|q\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t}\delta(q - q'). \tag{4}$$

To find the action of $\exp(-\mathrm{i}\hat{H}t)$ on the Berezin delta function, we make use of the fact that the GO Hamiltonian displays an SO(2, 1) dynamical or non-invariance group [6]. This can be seen if we decompose $\hat{H}$ as a linear combination of the self-conjugate operators

$$T_1 = -\tfrac{1}{2}\hat{p}_a C^{-1}_{ab} \hat{p}_b \qquad T_3 = \tfrac{1}{4}\hat{q}_a C_{ab} \hat{q}_b \tag{5}$$

which obey the commutation relation

$$[T_1, T_3]_- = -\mathrm{i}\tfrac{1}{4}(\hat{q} \cdot \hat{p} - \hat{p} \cdot \hat{q}) \equiv -\mathrm{i}T_2 \tag{6}$$

and also

$$[T_1, T_2]_- = -\mathrm{i}T_1 \qquad [T_3, T_2]_- = \mathrm{i}T_3. \tag{7}$$

The above commutation relations generate the SO(2, 1) Lie algebra. A more familiar form for this algebra is achieved by introducing the Γ_i operators

$$\Gamma_1 = \frac{1}{\sqrt{2}}(T_1 - T_3) \qquad \Gamma_2 = T_2 \qquad \Gamma_3 = \frac{1}{\sqrt{2}}(T_1 + T_3) \tag{8}$$

that obey

$$[\Gamma_i, \Gamma_j]_- = \mathrm{i}\varepsilon_{ijk} g^{kl} \Gamma_l \tag{9}$$

with the SO(2, 1) metric g^{ij} given by

$$g^{ij} = \mathrm{diag}(1, 1, -1). \tag{10}$$

The fact that the Hamiltonian is a linear combination of the generators of the SO(2, 1) algebra does not means that we have a SO(2, 1) symmetry but indicates that we can use the representations of the SO(2, 1) group to obtain the spectrum and eigenstates for the problem at hand [6]. In the following, we take an alternative route and use an SO(2, 1) Baker–Campbell–Hausdorff (BCH) formula to disentangle the expression for the propagator of the Schrödinger equation. It is interesting to note that the dynamical algebra of the N-dimensional GO is the same as for the N-dimensional oscillator [7]. To better understand this point, let us use the canonical anticommutation relations (1) in the T_2 generator

$$T_2 = \frac{1}{4}(\hat{q}\cdot\hat{p} - \hat{p}\cdot\hat{q}) = \frac{1}{2}\left(\mathrm{i}\frac{N}{2} + \hat{q}\cdot\hat{p}\right). \tag{11}$$

We can see that T_2 has a c-number term $\mathrm{i}\frac{N}{4}$ that can be identified as a central extension to the Poisson–Lie algebra due to the quantum conditions (1). The point is that this term happens to be proportional to the dimension N and has the opposite sign to the similar term that appears in the study of the HO dynamical group [7]. This fact give us a hint that the quantities obtained for the GO can be related to the equivalent ones for the HO by a shift $N \to -N$. It should be pointed out that besides the SO(2, 1) spectrum generating algebra, the GO has an SU($-N$) degeneracy group and also an obvious Sp(N) $\sim$ SO($-N$) geometrical invariance [4]. We now use the above SO(2, 1) Lie algebra to obtain the Green function for the GO. Using the generators T_i, we have that

$$\mathrm{e}^{-\mathrm{i}\hat{H}t}\delta(q-q') = \exp[-\mathrm{i}t(T_1 + 2\omega^2 T_3)]\delta(q-q'). \tag{12}$$

To find $\langle q', t|q, 0\rangle$, we have to disentangle the above expression in a product of exponential factors of the T_i. For this purpose, we need a BCH formula for the SO(2, 1) generators. A way of obtaining this formula is to write a faithful representation for the T_i in terms of the Pauli matrices σ_i and expand (12) in a Taylor series

$$T_1 = \frac{\sigma_1 - \mathrm{i}\sigma_2}{2\sqrt{2}} \qquad T_2 = \frac{-\mathrm{i}\sigma_3}{2} \qquad T_3 = \frac{\sigma_1 + \mathrm{i}\sigma_2}{2\sqrt{2}}. \tag{13}$$

Using the above in the equation

$$\exp[-\mathrm{i}t(T_1 + 2\omega^2 T_3)] = \mathrm{e}^{-\mathrm{i}xT_3}\mathrm{e}^{-\mathrm{i}yT_2}\mathrm{e}^{-\mathrm{i}zT_1} \tag{14}$$

we have that

$$x = 2\omega\tan(\omega t) \qquad y = 2\ln(\cos(\omega t)) \qquad z = \frac{1}{\omega}\tan(\omega t). \tag{15}$$

With the above BCH formula, we now verify the action of $\mathrm{e}^{-\mathrm{i}\hat{H}t}$ on $\delta(q-q')$. It is convenient to introduce a Berezin integral representation for the delta function

$$\delta(q-q') = \int \mathrm{d}^N p\, \mathrm{e}^{p_a(q-q')_a} \tag{16}$$

where the p_a are also elements of a Grassmann algebra.

It is now easy to verify that the action of each exponential factor in (14) on the delta function is given by

$$e^{-izT_1}\delta(q-q') = \int d^N p \, \exp\left[i\frac{z}{2}p_a C^{-1}_{ab} p_b + p_a(q-q')_a\right] \tag{17}$$

$$e^{-iyT_2} f(q) = e^{Ny/4} f(e^{-y/2}q) \tag{18}$$

where $f(q)$ is any function of q_a. With these results, the propagator reads

$$\langle q', t|q, 0\rangle = \int d^N p \, \exp\left\{-i\left[-\frac{z}{2}p_a C^{-1}_{ab} p_b + \frac{x}{4}q_a C_{ab} q_b + i\frac{Ny}{4} + ip_a(e^{-y/2}q - q')_a\right]\right\}. \tag{19}$$

If we now insert the values of x, y and z and appeal to the following Berezin integral identity [1]:

$$\int d^N p \, \exp(-p_a M_{ab} p_b + p_a q_a) = 2^{N/2}\sqrt{\det M}\,\exp(\tfrac{1}{4}q_a M^{-1}_{ab} q_b) \tag{20}$$

we get

$$\langle q', t|q, 0\rangle = \frac{1}{\sqrt{\det C}}\left[\frac{\sin(\omega t)}{i\omega}\right]^{N/2} e^{iS} \tag{21}$$

where

$$S = \frac{\omega}{2\sin(\omega t)} C_{ab}[(q_a q_b + q'_a q'_b)\cos(\omega t) - 2q_a q'_b]. \tag{22}$$

Thus, the propagator for the GO has the same phase as the HO, but with an opposite signal for N in the amplitude. This sign reversal of N can be regarded as a continuation $N \to -N$ in the result for the HO. In the next section, we get $\langle q', t|q, 0\rangle$ working in the Heisenberg picture.

4. The Dirac–Schwinger method

To obtain the Green function for the Schrödinger equation, Dirac [8] wrote (3) in the following way:

$$\langle q', t|q, 0\rangle = e^{iW(q',q;t)} \tag{23}$$

where $W(q', q; t)$ is a complex function of the end-point coordinates and time. It is easy to verify from (1) and (3) that this function is determined by the following relations:

$$-\frac{\partial W(q', q; t)}{\partial t} = \langle q', t|\hat{H}(\hat{q}(t), \hat{p}(t))|q, 0\rangle / \langle q', t|q, 0\rangle \tag{24}$$

$$\frac{\partial W(q', q; t)}{\partial q'_a} = \langle q', t|\hat{p}_a(t)|q, 0\rangle / \langle q', t|q, 0\rangle \tag{25}$$

$$-\frac{\partial W(q', q; t)}{\partial q_a} = \langle q', t|\hat{p}_a(0)|q, 0\rangle / \langle q', t|q, 0\rangle \tag{26}$$

$$W(q', q; 0) = -i\ln\delta(q' - q). \tag{27}$$

These equations surely have a flavour of the classical Hamilton–Jacobi method (HJ), so Dirac imagined $W(q', q; t)$ to be a matrix element between position eigenstates of a time-ordered operator $\hat{W}(\hat{q}(t), \hat{q}(0))$ that could be termed the 'quantum action operator', satisfying an operator version of the HJ equation. To solve for $\hat{W}(\hat{q}(t), \hat{q}(0))$, one must proceed as in the classical HJ method, replacing $\hat{p}(t)$ by the derivative of $\hat{W}(\hat{q}(t), \hat{q}(0))$ with respect to $\hat{q}(t)$ and so on, integrating the equations with the non-commutativity of the variables in mind [9].

In his classical paper on the effective interaction of quantum fields with a classical background, Schwinger [10] obtained an expression for the effective action in terms of the Green function of a non-relativistic quantum-mechanical problem. He followed Dirac to solve this problem, but instead of relying on the HJ method he noticed that the above equations relate the transition amplitude to the solution of the Heisenberg equations for $\hat{q}(t)$ and $\hat{p}(t)$

$$\frac{\mathrm{d}\hat{q}(t)}{\mathrm{d}t} = -\mathrm{i}[\hat{q}(t), \hat{H}]_- \qquad \frac{\mathrm{d}\hat{p}(t)}{\mathrm{d}t} = -\mathrm{i}[\hat{p}(t), \hat{H}]_-. \tag{28}$$

If we solve for $\hat{p}(t)$ in terms of $\hat{q}(t)$ and $\hat{q}(0)$ and insert this, in a time-ordered fashion, into (24)–(26), we are left with a set of first-order equations to integrate. For the GO, the equations in (28) are

$$\frac{\mathrm{d}\hat{q}_a(t)}{\mathrm{d}t} = (C^{-1})_{ab}\hat{p}_b(t) \qquad \frac{\mathrm{d}\hat{p}_a(t)}{\mathrm{d}t} = -\omega^2 C_{ab}\hat{q}^b(t) \tag{29}$$

with solutions

$$\hat{q}_a(t) = \hat{q}_a(0)\cos(\omega t) + \frac{1}{\omega}C^{-1}_{ab}\,\hat{p}_b(0)\sin(\omega t) \tag{30}$$

$$\hat{p}_a(t) = w\hat{q}_b(0)C_{ba}\sin(\omega t) + \hat{p}_a(0)\cos(\omega t). \tag{31}$$

As $\hat{H}$ is time-independent, $\hat{H}(\hat{q}(t), \hat{p}(t)) = H(\hat{q}(0), \hat{p}(0))$, only equation (30) is really needed to put $\hat{H}$ in a time-ordered form. Solving (30) for $\hat{p}(0)$ and inserting it into (2)

$$\hat{H} = \frac{1}{2}\left(\frac{\omega}{\sin(\omega t)}\right)^2 C_{ab}[\hat{q}_a(t)\hat{q}_b(t) + \hat{q}_a(0)\hat{q}_b(0)\cos^2(\omega t) - 2\hat{q}_a(t)\hat{q}_b(0)\cos(\omega t)$$
$$- \cos(\omega t)[\hat{q}_a(0), \hat{q}_b(t)]_+] + \tfrac{1}{2}\omega^2\hat{q}_a(0)C_{ab}\hat{q}_b(0). \tag{32}$$

The above anticommutator can be found using (30) and (1):

$$C_{ab}[\hat{q}_a(0), \hat{q}_b(t)]_+ = -\mathrm{i}N\frac{\sin(\omega t)}{\omega}. \tag{33}$$

Note that this is the negative of the corresponding commutator for the N-dimensional HO, a fact that is due to the canonical anticommutation relations (1) as we verified earlier in the realization of the SO(2, 1) generator T_2.

Now, if we define the expectation value

$$\langle\hat{H}\rangle \equiv \langle q', t|\hat{H}(\hat{q}(t), \hat{q}(0))|q, 0\rangle / \langle q', t|q, 0\rangle \tag{34}$$

we have that

$$\langle \hat{H} \rangle = \frac{\omega^2}{2} \csc^2(\omega t) C_{ab}[(q'_a q'_b + q_a q_b) - 2q'_a q_b \cos(\omega t)] + \mathrm{i}\frac{\omega N}{2} \cot(\omega t). \tag{35}$$

With the above result we are now in position to integrate (24)

$$W(q', q; t) = S(q', q; t) - \mathrm{i}\frac{N}{2} \ln \sin(\omega t) + \Phi(q', q) \tag{36}$$

where $S(q, q'; t)$ is given by (22) and $\Phi(q', q)$ is a time-independent function. In order to determine Φ, we substitute W in (25) and (26) to get

$$\frac{\partial \Phi(q', q)}{\partial q'_a} = \frac{\partial \Phi(q', q)}{\partial q_a} = 0. \tag{37}$$

By using the boundary condition (27), we have

$$\Phi = -\mathrm{i} \ln \left[(\mathrm{i}\omega)^{-N/2} \sqrt{\det C^{-1}} \right]. \tag{38}$$

So, the final answer for Dirac's 'quantum action' is given by

$$W(q', q; t) = S(q', q; t) - \mathrm{i} \ln \left\{ \frac{1}{\sqrt{\det C}} \left[\frac{\sin(\omega t)}{\mathrm{i}\omega} \right]^{N/2} \right\}. \tag{39}$$

As we can see, the real part of W is the classical action whilst its imaginary part is the pre-exponential factor as expected, since our system is quadratic [9], even if in a Grassmann sense.

5. Conclusions

In this paper we have explored the possibility of a Grassmann realization of non-relativistic quantum mechanics, computing the GO Green function by two different operator methods. In the first, we explored the fact that the Hamiltonian is a sum of SO(2, 1) Lie algebra generators, allowing one to use a BCH formula to disentangle the evolution operator as a product of exponential factors, each one with a simple realization in the configuration space and so avoiding the need for solving differential equations. Next, we turned to a method by Dirac and Schwinger that relates the solution of the Heisenberg equations for $\hat{q}(t)$ and $\hat{p}(t)$ to the transition amplitude by a simple integration of first-order differential equations. In both methods, we observed that the canonical anticommutation relations are responsible for the interpretation of the Grassmann coordinates as negative-dimensional degrees of freedom.

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